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On some questions about a family of cyclically presented groups ☆

Alberto Cavicchioli [a], E.A. O'Brien [b,*], Fulvia Spaggiari [a]

[a] *Dipartimento di Matematica, Università di Modena e Reggio E., Via Campi 213/B, 41100 Modena, Italy*
[b] *Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand*

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ABSTRACT

We study various questions about the generalised Fibonacci groups, a family of cyclically presented groups, which includes as special cases the Fibonacci, Sieradski, and Gilbert–Howie groups.

1. Introduction

Consider the class of groups with *cyclic presentation*:

$$G_n(w) = \langle x_1, \ldots, x_n \colon\ w = 1,\ \theta(w) = 1, \ldots, \theta^{n-1}(w) = 1 \rangle$$

where w is a reduced word in the alphabet $X = \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$ and θ is the automorphism of the free group of rank n defined by setting $\theta(x_i) = x_{i+1} \bmod n$. One of the motivations for the study of these groups is their connection with the topology of closed connected orientable 3-manifolds; see, for example, [5,12].

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* Corresponding author.
E-mail addresses: cavicchioli.alberto@unimo.it (A. Cavicchioli), e.obrien@auckland.ac.nz (E.A. O'Brien), spaggiari.fulvia@unimo.it (F. Spaggiari).

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If $w = x_i x_{i+m} x_{i+k}^{-1}$, then we obtain the *generalised Fibonacci groups* introduced in [4]:

$$G_n(m,k) = \langle x_1, \dots, x_n \colon\ x_i x_{i+m} = x_{i+k}\ (i = 1, \dots, n) \rangle$$

where the subscripts are taken modulo n.

For particular choices of parameters, these groups are well known: $G_n(1,2)$ are the *Fibonacci groups* $F(2,n)$ (see [7,17]); $G_n(2,1)$ are the *Sieradski groups* $S(n)$ (see [16,18]); $G_n(m,1)$ are the *Gilbert–Howie groups* $H(n,m)$ (see [9]).

We can immediately restrict our attention to those groups $G_n(m,k)$ whose parameters satisfy the conditions $0 < m < k < n$ and $(n,m,k) = 1$. Such groups are *irreducible*. Bardakov and Vesnin [2] prove:

- if $G_n(m,k)$ is not irreducible, then it is either trivial, cyclic, or a free product of $G_{n'}(m',k')$ for smaller values of n', m', k';
- if $G_n(m,k)$ is irreducible and either $(n,k) = 1$ or $(n, k-m) = 1$, then $G_n(m,k)$ is isomorphic to $G_n(t,1) = H(n,t)$, where $tk \equiv m \bmod n$ or $t(k-m) \equiv (n-m) \bmod n$ respectively.

This motivates the following definition in [2]: $G_n(m,k)$ is *strongly irreducible* if it is irreducible and $(n,k) > 1$ and $(n,k-m) > 1$.

Bardakov and Vesnin [2] pose, and study, a number of questions about these groups. These include:

- Under what conditions is $G_n(m,k)$ aspherical? Finite and non-trivial?
- Determine the number of isomorphism types among $G_n(m,k)$.
- Determine the structure of the largest abelian quotient, $A_n(m,k)$, of $G_n(m,k)$.
- Under what conditions is $G_n(m,k)$ the fundamental group of a 3-orbifold (in particular, a hyperbolic closed 3-manifold) of finite volume?

We summarise recent progress in answering these questions.

With a few exceptions, Gilbert and Howie [9] identify those $H(n,m)$ which are aspherical or finite. Williams [19] proves that a strongly irreducible group $G_n(m,k)$ is not aspherical if and only if $(m,k) = 1$ and either $n = 2k$, or $n = 2(k-m)$. He determines sufficient conditions for an irreducible group to be perfect. If, as he conjectures, these are also necessary, then every strongly irreducible group is not perfect; and he describes the structure of those which are finite and non-trivial. We show that $H(9,3)$ is infinite, thus reducing the undecided cases among irreducible (but not strongly irreducible) groups to 2.

Let $f(n)$ denote the number of isomorphism types among the irreducible groups $G_n(m,k)$. We obtain some new isomorphisms, and demonstrate that the known isomorphisms suffice to obtain $f(n)$ for all but four values of $n \leqslant 27$. We formulate a sharp conjecture for $f(p^\ell)$ where p is a prime.

Under the hypothesis of irreducibility, Corollary 5.8 of [5] shows that $A_n(m,k)$ is infinite if and only if $n \equiv 0 \bmod 6$, $m + k \equiv 3 \bmod 6$, and m is even. An equivalent result appears in [19, Theorem 4]. If $2k \equiv m \bmod n$, then we obtain a complete description of $A_n(m,k)$.

Corollary 3.5 of [5] is a slight improvement on [2, Theorem 3.1]: if n is odd and $(2k - m, n) = 1$, then $G_n(m,k)$ cannot be the fundamental group of a hyperbolic closed 3-orbifold of finite volume. If $G_n(m,k)$ is irreducible and $2k \equiv m \bmod n$, then we show that $G_n(m,k) \cong S(n)$, the fundamental group of a closed connected orientable 3-manifold. Finally, we prove that the split extension of an irreducible $G_n(m,k)$ by a cyclic group of order n has a homomorphism onto a particular triangle group if both $(n,k) = 1$ and $2(2k-m) \equiv 0 \bmod n$.

2. The isomorphism problem

The most general result on isomorphism is the following [2, Theorem 1.1].

Theorem 1. *Let $G_n(m,k)$ and $G_n(m',k')$ be irreducible groups. Assume that k' is divisible by $r=(n,k-m)$, $(n,k')=1$, and there exist integers $i\in\{1,\dots,r\}$ and $j\in\{1,\dots,n/r\}$ such that*

$$\begin{cases} i+j(k-m)\equiv(1-m) \bmod n, \\ m'+1\equiv(i+jk') \bmod n. \end{cases}$$

Then $G_n(m,k)\cong G_n(m',k')$.

Observe that the extra condition, $(n,k')=1$, omitted from the original statement is both necessary and a consequence of the proof: for example, $\mathbb{Z}_7\cong G_6(1,3)\ncong G_6(3,4)\cong\mathbb{Z}_2^3\rtimes\mathbb{Z}_7$.

Theorem 1 assumes both that k' is divisible by $(n,k-m)$ and $(n,k')=1$, so $r=1$. Hence, as was pointed out by the referee, we obtain an equivalent and simpler formulation.

Theorem 2. *Let $G_n(m,k)$ and $G_n(m',k')$ be irreducible groups and assume $(n,k')=1$. If $m'(m-k)\equiv mk' \bmod n$, then $G_n(m,k)$ is isomorphic to $G_n(m',k')$.*

We record some obvious consequences.

Corollary 3.

(1) *If $n\geqslant 5$ is odd, then $G_n(n-3,n-1)\cong G_n(n-3,n-2)$.*
(2) *$G_{2h+1}(h,h+1)\cong G_{2h+1}(h,2h)\cong G_{2h+1}(1,2)=F(2,2h+1)$.*
(3) *If $(2h+1,k-1)=1$, then $G_{2h+1}(1,k)\cong G_{2h+1}(1,2h+2-k)$.*

Proof. We illustrate the method by proving (3). By hypothesis, $(2h+1,k-1)=1$ and so $(2h+1, 2h+2-k)=1$. Since $(1-k)\equiv(2h+2-k) \bmod (2h+1)$, the result follows. □

Corollary 4. *If there exists β such that $\beta s\equiv 1 \bmod n$ and $\beta(1-t)\equiv 1 \bmod n$, then $G_n(1,t)\cong G_n(1,s)$.*

Proof. Since $\beta s\equiv 1 \bmod n$, we conclude that $(n,s)=1$. □

Proposition 5. *If $(n,m)=1$, then $G_n(m,k)$ is isomorphic to $G_n(1,t)$, where $tm\equiv k \bmod n$.*

Proof. We rename the generators of $G_n(m,k)$: $c_1=x_1, c_2=x_{1+m},\dots,c_n=x_{1+(n-1)m}$. The first relation $x_1x_{1+m}=x_{1+k}$ of $G_n(m,k)$ becomes $c_1c_2=c_{1+t}$, where $c_{1+t}=x_{1+tm}=x_{1+k}$ with $tm\equiv k \pmod n$. The next relation $c_2c_3=c_{2+t}$ corresponds to $x_{1+m}x_{1+2m}=x_{1+m+k}$ since $c_{2+t}=x_{1+(1+t)m}=x_{1+m+k}$. Similarly, $c_jc_{j+1}=c_{j+t}$ corresponds to $x_{1+(j-1)m}x_{1+jm}=x_{1+(j+t-1)m}$; that is, $x_{1+(j-1)m}x_{1+jm}=x_{1+(j-1)m+k}$. If j runs over $\{1,\dots,n\}$, then $1+(j-1)m$, taken mod n, runs over the same set. Therefore $G_n(m,k)\cong G_n(1,t)$ where $tm\equiv k \bmod n$. □

Proposition 6.

(1) $G_n(m,k)\cong G_n(m,n+m-k)\cong G_n(n-m,n-m+k)$.
(2) *If $(n,t)=1$, then $G_n(m,k)\cong G_n(mt,kt)$.*
(3) $G_{2h}(2h-1,2h-2)\cong G_{2h}(2h-1,1)\cong G_{2h}(1,2h-1)\cong G_{2h}(1,2)=F(2,2h)$.

Proof. (1) Taking the inverse relation of $x_ix_{i+m}=x_{i+k}$ and substituting i with $-i-m$, we get $x_{-i}^{-1}x_{-(i+m)}^{-1}=x_{-(i+m-k)}^{-1}$. Setting $y_i:=x_{-i}^{-1}$ yields the relation $y_iy_{i+m}=y_{i+n+m-k}$ which defines $G_n(m,n+m-k)$. The second isomorphism, which appears as [2, Lemma 1.1], can be similarly established.

(2) Set $G = G_n(m,k) = \langle x_i\colon\ x_i x_{i+m} = x_{i+k}\rangle$ and $H = G_n(mt,kt) = \langle y_i\colon\ y_i y_{i+mt} = y_{i+kt}\rangle$. Let $\phi : G \to H$ be defined by setting $\phi(x_j) = y_{1+t(j-1)}$. The map ϕ is onto since $(n,t) = 1$. Furthermore, ϕ sends the defining relations of G to those of H:

$$\phi\bigl(x_i x_{i+m} x_{i+k}^{-1}\bigr) = y_{1+t(i-1)} y_{1+t(i+m-1)} y_{1+t(i+k-1)}^{-1} = y_j y_{j+mt} y_{j+kt}^{-1}$$

where $j = 1 + t(i-1)$. Thus ϕ is a homomorphism and, since it is invertible, it is an isomorphism.

(3) This follows from (1). □

We illustrate the previous results by identifying some isomorphisms among $G_{27}(m,k)$. Corollary 3 implies that $G_{27}(24,26) \cong G_{27}(24,25)$, $G_{27}(13,14) \cong G_{27}(13,26) \cong G_{27}(1,2) \cong F(2,27)$, and $G_{27}(1,18) \cong G_{27}(1,10)$. Corollary 4 implies that $G_{27}(1,11) \cong G_{27}(1,17)$. Proposition 5 implies that $G_{27}(2,5) \cong G_{27}(1,16)$. Proposition 6 implies that $G_{27}(2,5) \cong G_{27}(2,24) \cong G_{27}(25,3)$ and $G_{27}(2,5) \cong G_{27}(4,10) \cong G_{27}(8,20) \cong G_{27}(10,25) \cong G_{27}(14,8)$.

Proposition 7. *If p is an odd prime, then there are at most $(p-1)/2$ isomorphism types among the irreducible groups $G_p(m,k)$.*

Proof. If p is prime, then $(p,m) = 1$. Proposition 5 implies that $G_p(m,k) \cong G_p(1,t)$ for some $t \in \{2,\dots,p-1\}$, where $tm \equiv k \bmod p$. Since $(p, t-1) = 1$, there exists β such that $\beta(1-t) \equiv 1 \bmod p$.

If $2 \leqslant t \leqslant (p+1)/2$, then $s = p+1-t$ satisfies $(p+1)/2 \leqslant s \leqslant p-1$. Corollary 4 now implies that $G_p(1,t) \cong G_p(1,s)$ since $\beta s = \beta(p+1-t) \equiv 1 \bmod p$. Hence the isomorphism types arise by choosing $t \in \{2,\dots,(p+1)/2\}$, and so $f(p) \leqslant (p-1)/2$. □

Our investigations, reported in Section 5, suggest the following stronger result.

Conjecture 8. *If $n = p^\ell$ for an odd prime p and positive integer ℓ, then $f(n) = p^\ell - \frac{(p-1)}{2} p^{(\ell-1)} - 1$. If $\ell > 2$, then $f(2^\ell) = 3(2^{\ell-2})$.*

3. The abelianisation of $G_n(m,k)$

We obtain a complete description of the abelianisation of $G_n(m,k)$ when $2k \equiv m \bmod n$, and so extend [19, Lemma 5].

Lemma 9. *Assume $2k \equiv m \bmod n$.*

- *$G_n(m,k)$ is perfect if and only if $n/(n,k) \equiv \pm 1 \bmod 6$.*
- *The abelianisation of $G_n(m,k)$ is isomorphic to $\mathbb{Z}^{2(n,k)}$, $\mathbb{Z}_3^{(n,k)}$, or $\mathbb{Z}_2^{2(n,k)}$ if and only if $n/(n,k) \equiv 0$, $n/(n,k) \equiv \pm 2$, or $n/(n,k) \equiv \pm 3 \bmod 6$ respectively.*

Proof. If $G_n(m,k)$ is irreducible, then $2k \equiv m \bmod n$ implies that $(n,k) = 1$. Proposition 6(2) implies that $G_n(m,k) \cong G_n(2k,k) \cong G_n(2,1) = S(n)$. Recall from [6, Theorem 2.1] that $S(n) \cong \pi_1(M_n)$, where M_n is the n-fold cyclic cover of the 3-sphere, branched over the trefoil knot. Thus M_n is the Brieskorn manifold of [13] and its abelianisation is well known—see, for example, [15, p. 304].

If $G_n(2k,k)$ is not irreducible, then [2, Lemma 1.2] shows that $G_n(2k,k)$ is isomorphic to a free product of (n,k) copies of $G_{n/(n,k)}(2k/(n,k), k/(n,k))$, which is irreducible. Hence the abelianisation of $G_n(2k,k)$ is trivial, $\mathbb{Z}^{2(n,k)}$, $\mathbb{Z}_3^{(n,k)}$, or $\mathbb{Z}_2^{2(n,k)}$ according to the stated congruence conditions. □

In summary, Proposition 6(2), [6, Corollary 2.2] and [18, Theorems B–C] imply the following: if $G_n(m,k)$ is irreducible and $2k \equiv m \bmod n$, then $G_n(m,k)$ is infinite if and only if $n \geqslant 6$, and it has a free subgroup of rank 2 when $n \geqslant 7$.

We now briefly discuss $A_n(1,t)$ for arbitrary n. It is well known that $A_n(1,2)$ is finite, of order $L_n - 1 - (-1)^n$, where L_n is the nth Lucas number (see, for example, [11, Chapter 6]).

Proposition 10. *The structure of $A_n(1,t)$, where $t \in \{2,\dots,n-1\}$, is determined by the diagonal form of the integral $t\times t$ matrix*

$$\begin{pmatrix} a_{n+1}-1 & a_{n+t} & a_{n+t-1} & \dots & a_{n+3} & a_{n+2} \\ a_{n+2} & a_{n+t+1}-1 & a_{n+t} & \dots & a_{n+4} & a_{n+3} \\ a_{n+3} & a_{n+t+2} & a_{n+t+1}-1 & \dots & a_{n+5} & a_{n+4} \\ \vdots & \dots & \dots & \dots & \vdots & \vdots \\ a_{n+t-1} & a_{n+2t-2} & a_{n+2t-3} & \dots & a_{n+t+1}-1 & a_{n+t} \\ a_n+1 & a_{n+t-1} & a_{n+t-2} & \dots & a_{n+2} & a_{n+1}-1 \end{pmatrix}$$

where $a_i + a_{i+1} = a_{i+t}$ $(i \geqslant 1)$ and $a_1 = 1, a_i = 0$ $(2 \leqslant i \leqslant t)$.

Proof. We sketch a proof. The generators, $x_1,\dots,x_n$, of $A_n(1,t)$ commute and satisfy the following relations:

$$\begin{aligned} x_1x_2 &= x_{t+1}, \\ x_2x_3 &= x_{t+2}, \\ x_3x_4 &= x_{t+3}, \\ &\vdots \\ x_{t-1}x_t &= x_{2t-1}, \\ x_tx_{t+1} &= x_{2t}, \\ x_{t+1}x_{t+2} &= x_{2t+1}, \\ x_{t+2}x_{t+3} &= x_{2t+2}, \\ &\vdots \\ x_{n-t}x_{n-t+1} &= x_n. \end{aligned}$$

Hence $\{x_1,\dots,x_t\}$ generate $A_n(1,t)$, and $x_i = x_1^{a_i}x_2^{b_i^2}\cdots x_t^{b_i^t}$ for $i>t$.

We use the relations $x_i = x_{i-t}x_{i-t+1}$ and those implied by commutativity to deduce that

$$\begin{aligned} a_i &= a_{i-t} + a_{i-t+1}, \quad i>t, \\ b_i^j &= b_{i-t}^j + b_{i-t+1}^j, \quad 2 \leqslant j \leqslant t, \end{aligned}$$

where $a_1 = 1, a_i = 0, b_i^j = \delta_{i,j}$ for $2 \leqslant i, j \leqslant t$. Thus $b_i^j = a_{i+t-j+1}$ for all $i \geqslant 1$.

Hence the structure of $A_n(1,t)$ can be deduced from the diagonal form of the $t\times t$ matrix:

$$\begin{pmatrix} a_{n-t+1}+a_{n-t+2}-1 & b_{n-t+1}^2+b_{n-t+2}^2 & b_{n-t+1}^3+b_{n-t+2}^3 & \cdots & b_{n-t+1}^t+b_{n-t+2}^t \\ a_{n-t+2}+a_{n-t+3} & b_{n-t+2}^2+b_{n-t+3}^2-1 & b_{n-t+2}^3+b_{n-t+3}^3 & \cdots & b_{n-t+2}^t+b_{n-t+3}^t \\ a_{n-t+3}+a_{n-t+4} & b_{n-t+3}^2+b_{n-t+4}^2 & b_{n-t+3}^3+b_{n-t+4}^3-1 & \cdots & b_{n-t+3}^t+b_{n-t+4}^t \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{n-1}+a_n & b_{n-1}^2+b_n^2 & b_{n-1}^3+b_n^3 & \cdots & b_{n-1}^t+b_n^t \\ a_n+1 & b_n^2 & b_n^3 & \cdots & b_n^t-1 \end{pmatrix}.$$

The result now follows readily. $\square$

Of course, the isomorphism type of $G_n(1,t)$ is not determined by its abelian quotient invariants: $A_{11}(1,3) \cong A_{11}(1,4) \cong \mathbb{Z}_{23}$ but $G_{11}(1,3) \ncong G_{11}(1,4)$ (since their derived groups have abelian quotient invariants 2^{11} and 3^{11} respectively).

4. Split extensions

Let $E_n(m,k)$ denote the split extension of $G_n(m,k)$ by $\mathbb{Z}_n = \langle \theta\colon\ \theta^n = 1\rangle$, where θ is the automorphism sending each generator x_i to x_{i+1} (subscripts taken modulo n). The relations $x_i x_{i+m} = x_{i+k}$ of $G_n(m,k)$ imply

$$x\theta^{-m}x\theta^m = \theta^{-k}x\theta^k$$

where $x := x_n$, and $x_i = \theta^{-i}x\theta^i$. Setting $y = \theta^m x^{-1}$ (and eliminating $x = y^{-1}\theta^m$) yields

$$E_n(m,k) = \langle \theta, y\colon\ \theta^n = 1,\ \theta^{k-m}y^2 = y\theta^k\rangle.$$

Assume $2k \equiv m \bmod n$. As we observed in Lemma 9, if $G_n(m,k)$ is irreducible, then it is isomorphic to $G_n(2,1) = S(n)$. Further $E_n(m,k)$ is isomorphic to the fundamental group of the 3-dimensional orbifold whose underlying space is the 3-sphere and whose singular set is the trefoil knot with branching index n (see for example [6, Theorem 2.1]).

Lemma 11. *If $n \geqslant 3$ is odd, then $G_n(1,(n+1)/2) \cong S(n)$ is isomorphic to the derived group of the centrally extended triangle group*

$$\Gamma = \langle \gamma_1, \gamma_2, \gamma_3\colon\ \gamma_1^n = \gamma_2^2 = \gamma_3^3 = \gamma_1\gamma_2\gamma_3\rangle.$$

If $n \geqslant 7$ is odd, then the centre of $G_n(1,(n+1)/2)$ is $\mathbb{Z}$, otherwise it is $\mathbb{Z}_2$. If $p \geqslant 5$ is a prime, there is a homomorphism from $G_p(1,(p+1)/2)$ onto $\mathrm{SL}(2,p)$. *Furthermore, $G_5(1,3) \cong \mathrm{SL}(2,5)$ and $G_3(1,2) \cong Q_8$.*

Proof. The first two assertions follow from [13, Section 3] since $G_n(1,(n+1)/2)$ is isomorphic to the fundamental group of the Brieskorn manifold $M(n,2,3)$. To prove the third, we define a map from $E_p(1,(p+1)/2) \to \mathrm{SL}(2,p)$:

$$\theta \to A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad y \to B = \begin{pmatrix} 0 & -2 \\ (p+1)/2 & 1 \end{pmatrix}.$$

One can easily verify that this is an epimorphism which induces an epimorphism from $G_p(1,(p+1)/2)$ to $\mathrm{SL}(2,p)$, sending $x_i \longmapsto A^iB^{-1}A^{i+1}$. The last follows from [9]. $\square$

Theorem 12. *Let $G_n(m,k)$ be irreducible.*

(a) *If $2(2k-m) \equiv 0 \bmod n$, then $E_n(m,k)$ has a homomorphism onto the subgroup of* $\mathrm{SL}(2,\mathbb{C})$ *having presentation*

$$\{A, B\colon\ A^n = B^3 = 1,\ A^{2k-m} = (BA^k)^2\}.$$

(b) *If $(n,k) = 1$, then $E_n(m,k)$ has a homomorphism onto the group defined by the presentation $\{u, v\colon\ v^n = 1, (uv)^3 = 1, v^{-\eta(2k-m)} = u^2\}$ where $\eta k \equiv 1 \bmod n$, for some integer η.*

(c) *If $(n,k) = 1$ and $2(2k-m) \equiv 0 \bmod n$, then $E_n(m,k)$ covers the triangle group of type $(n,2,3)$ if n is odd, and of type $((n,2k-m),2,3)$ if n is even. If $(n,k) = 1$, $2(2k-m) \equiv 0 \bmod n$ and $(n,2k-m) \geqslant 6$, then $G_n(m,k)$ is infinite.*

Proof. Recall $E_n(m,k) = \langle \theta, y\colon\ \theta^n = 1, \theta^{k-m}y^2 = y\theta^k\rangle$.

(a) We will exhibit a homomorphism $E_n(m,k) \to \mathrm{SL}(2,\mathbb{C})$ which both satisfies the relations of $E_n(m,k)$ and sends

$$\theta \to A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad y \to B = \begin{pmatrix} \alpha & \beta \\ 1 & \gamma \end{pmatrix}$$

where $\lambda^n = 1$, $\beta \neq 0$, and $\alpha\gamma - \beta = 1$. Such a homomorphism implies that

$$\begin{pmatrix} \alpha & \beta \\ 1 & \gamma \end{pmatrix}^2 = \begin{pmatrix} \lambda^{m-k} & 0 \\ 0 & \lambda^{k-m} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 1 & \gamma \end{pmatrix} \begin{pmatrix} \lambda^k & 0 \\ 0 & \lambda^{-k} \end{pmatrix}.$$

This gives the system of equations

$$\begin{cases} \alpha^2 - \alpha\lambda^m + \beta = 0, \\ \beta(\alpha+\gamma) = \beta\lambda^{m-2k}, \\ \alpha + \gamma = \lambda^{2k-m}, \\ \gamma^2 - \gamma\lambda^{-m} + \beta = 0, \\ \beta = \alpha\gamma - 1. \end{cases}$$

Since $\beta \neq 0$, the second and third equations imply $\lambda^{2(2k-m)} = 1$, which holds because n divides $2(2k-m)$. The system has the unique solution given by

$$\alpha = \frac{1}{\lambda^{m-2k} - \lambda^m}, \qquad \beta = \frac{\lambda^{2(m-k)} - \lambda^{2m} - 1}{(\lambda^{m-2k} - \lambda^m)^2}, \qquad \gamma = \frac{-\lambda^{2(m-k)}}{\lambda^{m-2k} - \lambda^m}.$$

Assume $\lambda^m(\lambda^{-2k} - 1) = 0$. Then $|\lambda^m||\lambda^{-2k} - 1| = 0$, and so $2k \equiv 0 \pmod n$. But $G_n(m,k)$ is irreducible and so $0 < m < k < n$. Since $2(2k-m) \equiv 0 \bmod n$, we deduce that $2m \equiv 0 \bmod n$, a contradiction. Hence $\lambda^m(\lambda^{-2k} - 1) \neq 0$.

Let $\tau(B)$ be the square of the trace of the matrix B. Then

$$\tau(B) = \frac{(1 - \lambda^{2(m-k)})^2}{(\lambda^{m-2k} - \lambda^m)^2} = \frac{1 + \lambda^{4(m-k)} - 2\lambda^{2(m-k)}}{1 + \lambda^{2m} - 2\lambda^{2(m-k)}} = 1$$

since $\lambda^{2(m-2k)} = 1$ and $\lambda^{4(m-k)} = \lambda^{2m}$ as $2(2k-m) \equiv 0 \bmod n$. Hence B is elliptic. By [1, p. 39], we determine the multiplier M^2 of B by applying the quadratic formula

$$M^2 = \frac{1}{2}\Big[\tau(B) - 2 \pm \sqrt{-4\tau(B) + \tau^2(B)}\Big].$$

Since $M^2 = (-1 \pm i\sqrt{3})/2$, we conclude that B has order 3. The statement follows.

(b) If $y^3 = 1$, then the relation $\theta^{k-m}y^2 = y\theta^k$ becomes $\theta^{k-m} = y\theta^k y$, hence $\theta^{2k-m} = (y\theta^k)^2$. Thus adding the relation $y^3 = 1$ gives a homomorphism from $E_n(m,k)$ onto $\langle \theta, y\colon\ \theta^n = y^3 = 1, \theta^{2k-m} = (y\theta^k)^2\rangle$. If $(n,k) = 1$, then there exist integers ξ and η such that $\xi n + \eta k = 1$. Setting $u = y\theta^k$ and $v = \theta^{-k}$, we deduce that $E_n(m,k)$ covers the group defined in (b).

(c) If n is odd, then by (b) $E_n(m,k)$ covers $\langle v, u\colon\ v^n = u^2 = (uv)^3 = 1\rangle$. If n is even, then the relation $v^{(n,2k-m)} = 1$ implies a homomorphism of $E_n(m,k)$ onto the triangle group of type $((n,2k-m),2,3)$. The infiniteness claim now follows from [8, §6.4]. □

Consider the case when $(n,k) = 1$ and $2(2k-m) \equiv 0 \bmod n$. If n is also odd, then $2k \equiv m \bmod n$; since $G_n(m,k) \cong S(n)$, it is infinite for $n \geqslant 6$. If n is even, then (c) has new consequences: for example, it implies that $G_{12}(4,5)$ is infinite.

5. Investigating $G_n(m,k)$ for small values of n

We investigated the irreducible groups $G_n(m,k)$ for values of $n \leqslant 27$. We used implementations in MAGMA [3] of algorithms to perform coset enumerations, compute abelian quotient invariants and (normal) subgroups of low index, and construct presentations for subgroups and p-quotients of finitely-presented groups. We refer the interested reader to [10, Chapters 5 and 9] for details and references to these algorithms.

5.1. Isomorphism

We sought to solve the isomorphism problem among the irreducible $G_n(m,k)$ for small values of n. We applied the isomorphisms identified in Theorem 2, its corollaries, and Propositions 5–6 to obtain both an upper bound $U(n)$ to the value of $f(n)$, and a potentially redundant list of isomorphism types. We then used invariants of groups in the resulting list to obtain a lower bound $L(n)$ to the value of $f(n)$. These bounds frequently coincided, so allowing us to deduce the precise value of $f(n)$.

In most cases, it sufficed to compute the abelian quotient invariants of a group and those of its derived group to distinguish it from any other on the list. We note the exceptional cases.

- We proved that $G_{14}(1,3)$ is not isomorphic to $G_{14}(1,5)$ by showing that, among their normal subgroups of index 16, the number of distinct abelian quotient invariants is 8 and 9 respectively.
- The p-class 2 241-quotient of the derived group of $G_{22}(1,5)$ has order 241^{22}; the corresponding quotient of the derived group of $G_{22}(1,7)$ has order 241^{44}.
- PSL(2, 5) is a homomorphic image of $G_{25}(1,3)$ but not of $G_{25}(1,6)$.
- $G_{26}(1,13)$ is finite, $G_{26}(13,14)$ is infinite.

We summarise our results in Table 1. For $n \in \{3,\dots,27\}$, we record the values of $L(n)$ and $U(n)$; for each of the $U(n)$ groups, we list one defining value of the parameters (m,k). For $n \in \{17,19,21,23\}$, the values of $L(n)$ and $U(n)$ differ by 1. The unresolved cases are listed in Table 2.

Table 1 demonstrates that Conjecture 8 is sharp. For $n \in \{28,\dots,200\}$, we computed $U(n)$ and counted the number of distinct abelian quotient invariants among $G_n(m,k)$. This provided additional evidence for the correctness of Conjecture 8; it also suggests that there is at most one coincidence among the values of the abelian quotient invariants of $G_n(m,k)$ when $n = p^\ell$.

5.2. Finiteness

We summarise the results of Gilbert and Howie [9] and Williams [19], with known isomorphisms applied.

Theorem 13.

(i) *Suppose* $(n,m) \notin \{(8,3),(9,3),(9,4),(9,7)\}$. *Then* $H(n,m)$ *is finite if and only if* $m = 0$ *or* 1, *or* $(n,m) = (2\ell, \ell+1)$ *where* $\ell \geqslant 1$, *or*

$$(n,m) \in \{(3,2),(4,2),(5,2),(5,3),(6,3),(7,4)\}.$$

(ii) *Let* $G = G_n(m,k)$ *be strongly irreducible and assume* $G \neq 1$. *Then* G *is finite if and only if* $(m,k) = 1$ *and* $n = 2k$ *or* $n = 2(k-m)$, *in which case* $G \cong \mathbb{Z}_s$ *where* $s = 2^{n/2} - (-1)^{m+(n/2)}$.

The structure of (the then known) finite irreducible groups among $H(n,m)$ is recorded in [9, Table 1]. Some of the exceptions from Theorem 13(i) have since been resolved. We now know that $H(8,3) \cong G_8(3,1)$ is a soluble group of order $3^{10} \cdot 5$ and derived length 3. First established by R.M. Thomas, its order can now be determined by a routine coset enumeration in MAGMA.

We now prove that $H(9,3)$ is infinite. Recall first Newman's extension [14] of the Golod–Šafarevič theorem, which we summarise for the prime 2.

Table 1
Lower and upper bounds for $f(n)$ for $n \leqslant 27$

n	$L(n)$	$U(n)$	Parameters (m,k)
3	1	1	$(1,2)$
4	2	2	$(1,2)$, $(2,3)$
5	2	2	$(1,k)$ $k\in\{2,3\}$
6	5	5	$(1,k)$ $k\in\{2,3\}$, $(2,3)$, $(3,4)$, $(4,5)$
7	3	3	$(1,k)$ $k\in\{2,3,4\}$
8	6	6	$(1,k)$ $k\in\{2,3,4\}$, $(2,3)$, $(2,5)$, $(4,5)$
9	5	5	$(1,k)$ $k\in\{2,\dots,5\}$, $(3,4)$
10	8	8	$(1,k)$ $k\in\{2,\dots,5\}$, $(2,k)$ $k\in\{3,5\}$, $(4,7)$, $(5,6)$
11	5	5	$(1,k)$ $k\in\{2,\dots,6\}$
12	12	12	$(1,k)$ $k\in\{2,\dots,6\}$, $(2,k)$ $k\in\{3,7\}$, $(3,k)$ $k\in\{4,5\}$ $(4,k)$ $k\in\{5,7\}$, $(6,7)$
13	6	6	$(1,k)$ $k\in\{2,\dots,7\}$
14	11	11	$(1,k)$ $k\in\{2,\dots,7\}$, $(2,k)$ $k\in\{3,5,7\}$, $(4,9)$, $(7,8)$
15	12	12	$(1,k)$ $k\in\{2,\dots,8\}$, $(3,k)$ $k\in\{4,5,7\}$, $(5,6)$, $(5,7)$
16	12	12	$(1,k)$ $k\in\{2,\dots,8\}$, $(2,k)$ $k\in\{3,5,9\}$, $(4,5)$, $(8,9)$
17	7	8	$(1,k)$ $k\in\{2,\dots,9\}$
18	17	17	$(1,k)$ $k\in\{2,\dots,9\}$, $(2,k)$ $k\in\{3,5,7,9\}$, $(3,k)$ $k\in\{4,7\}$ $(4,11)$, $(6,7)$, $(9,10)$
19	8	9	$(1,k)$ $k\in\{2,\dots,10\}$
20	18	18	$(1,k)$ $k\in\{2,\dots,10\}$, $(2,k)$ $k\in\{3,5,11\}$ $(4,k)$ $k\in\{5,7,11\}$, $(5,6)$, $(5,8)$, $(10,11)$
21	15	16	$(1,k)$ $k\in\{2,\dots,11\}$, $(3,k)$ $k\in\{4,5,7,8\}$, $(7,8)$, $(7,9)$
22	17	17	$(1,k)$ $k\in\{2,\dots,11\}$, $(2,k)$ $k\in\{3,5,7,9,11\}$, $(4,13)$, $(11,12)$
23	10	11	$(1,k)$ $k\in\{2,\dots,12\}$
24	26	26	$(1,k)$ $k\in\{2,\dots,12\}$, $(2,k)$ $k\in\{3,5,7,13\}$, $(3,k)$ $k\in\{4,5,8,10\}$ $(4,k)$ $k\in\{5,7\}$, $(6,k)$ $k\in\{7,13\}$, $(8,k)$ $k\in\{9,13\}$, $(12,13)$
25	14	14	$(1,k)$ $k\in\{2,\dots,13\}$, $(5,6)$, $(5,7)$
26	20	20	$(1,k)$ $k\in\{2,\dots,13\}$, $(2,k)$ $k\in\{3,5,7,9,11,13\}$, $(4,15)$, $(13,14)$
27	17	17	$(1,k)$ $k\in\{2,\dots,14\}$, $(3,k)$ $k\in\{4,5,10\}$, $(9,10)$

Table 2
Possible isomorphisms

n	Parameters (m,k)
17	$(1,3)$, $(1,4)$
19	$(1,3)$, $(1,6)$
21	$(1,6)$, $(1,9)$
23	$(1,3)$, $(1,7)$

Theorem 14. *Let G be a group with a finite presentation on b generators and r relations. Let $G_1 := [G,G]G^2$ and $G_2 := [G_1,G]G_1^2$, where the elementary abelian 2-groups G/G_1 and G_1/G_2 have rank d and e respectively. If $r-b \leqslant d^2/2+d/2-d-e+(e-d/2-d^2/4)d/2$, then G is infinite.*

Lemma 15. *The group $H := H(9,3) \cong G_9(3,4)$ is infinite.*

Proof. The second derived group, K, of H has index 448 in H. We obtain, using a Reidemeister–Schreier rewriting procedure [10, §2.5], a presentation for K on 321 generators and 768 relations. Now K has abelian quotient invariants $2^{36}4^7$. Let Q denote its 2-quotient of p-class 2: Q has order 2^{604}, its Frattini quotient has rank $d=43$, and so $e=561$. Theorem 14 implies that K is infinite. □

The other exceptions, $H(9,4) \cong G_9(1,3)$ and $H(9,7) \cong G_9(1,4)$, remain unresolved.

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